澳門蔡氏教育文化基金會主辦

新華學校

ESCOLA SUN WAH

Card Tricks as a Context for Mathematical Modelling: A Task Sequence Design and Implementation

紙牌魔術走進數學建模的課堂：任務串驅動的課堂設計及實施

簡煥森
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摘要 Abstract

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聽牌術	
模運算、組合數學、……	引入數學課程

《排列組合》	
4課時，選修課, 澳門資助學校	高二，10名，自願參與

構建聽牌術的數學模型		
數列、排列組合	演算法設計	創造自己的魔術

實際教學效果	
數學建模素養量表	排列組合月考測試

Listening Card Trick(**LCT**)	
Modular, Combinatorics,……	Be used as a scenario for mathematical modeling（**MM**）

Permutation and Combination	
4 Lessons， Elective course, Aided school in Macau	10 Grade 11 Students， Voluntary

Constructing the LCT model		
Sequence, Permutation and Combination	Algorithm design	Create your own LCT

Actual Teaching Effect	
Mathematical modeling competency scale	Permutation and combination monthly test

1. Introduction

Mathematical Modeling(MM)

Represent, analyze, & solve real-world problems (Blum & Niss, 1991)	Important, mathematical thinking, creativity, communication, & problem-solving(English & Sriraman, 2010)

3 Challenges

張文剛 Zhang (2020)	董玉成，徐斌豔 Dong and Xu (2014)

(1) Textbooks : Lack Systematic and Practical MM Content & Cases

Do not fully reflect the **process** and **methods** of MM	Do not reflect the connection between MM and ⋯ **culture** and **art** ⋯	Lack **real situations**, ⋯ & **explanation** of the modeling process.

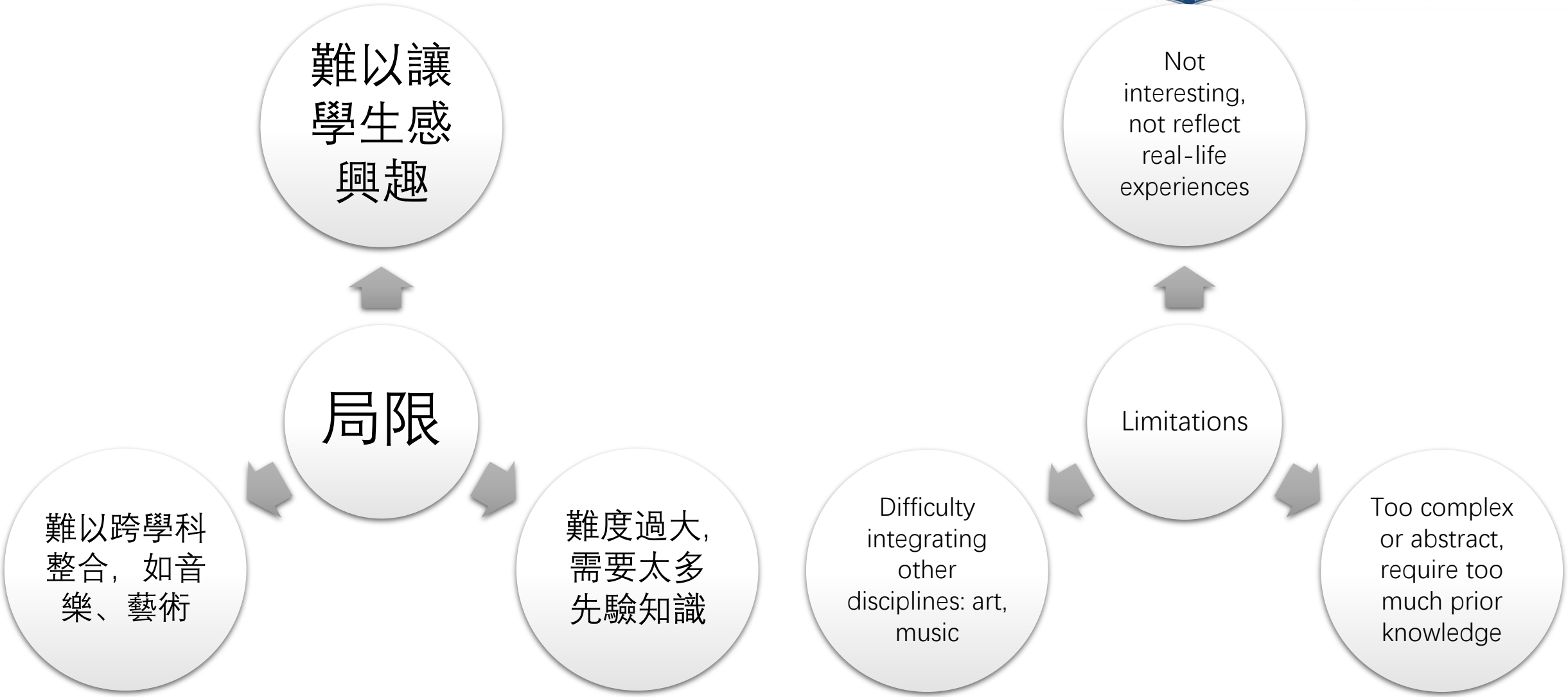
(2)Teachers: Lack deep understanding of MM

Eg.: Process of MM, teaching experience and methods	Difficult to organize students to **conduct** modeling activities

(3) Evaluation: Inconsistent with the modeling objectives

Only foucs on final results,	Ignoring thinking process ……

1. Introduction



張文剛 Zhang (2020)， 董玉成，徐斌豔 Dong and Xu (2014)

1. Introduction

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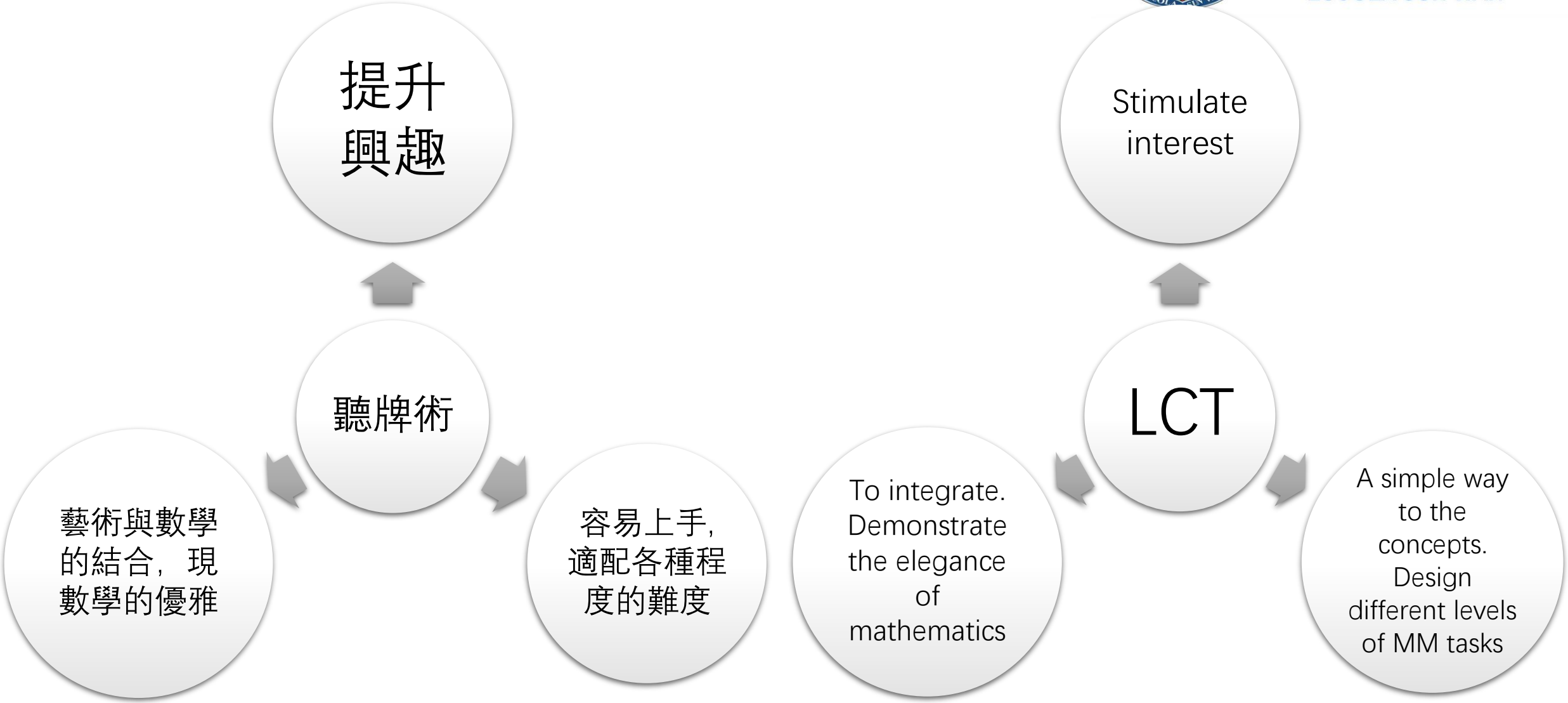

簡煥森 JIAN (2023)

- The research question of this study is:

How does using LCT as a scenario for mathematical modeling affect the mathematical modeling competence of high school students?

使用聽牌術作為數學建模的教學情境，
對高中生的數學建模能力有什麼影響？

- The research hypothesis of this study is:

Using LCT as a scenario for mathematical modeling can improve the mathematical modeling competence of high school students.

使用聽牌術作為數學建模的教學情境，
可以提高高中生的數學建模能力。

2. LITERATURE REVIEW

2.1 Si Stebbins stack (SSS) and Listening Card Trick (LCT): principles and applications

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Si Stebbins stack (SSS) → **Listening card trick (LCT)** → **Mathematical modeling (MM)**

Clubs	Hearts	Spades	Diamonds
Ace	4	7	10
King	3	6	9
Queen	2	5	8
Jack	Ace	4	7
10	King	3	6
9	Queen	2	5
8	Jack	Ace	4
7	10	King	3
6	9	Queen	2
5	8	Jack	Ace
4	7	10	King
3	6	9	Queen
2	5	8	Jack

Si Stebbins stack

Each card in a Si Stebbins stack alternates suit in the CHaSeD order (Clubs Hearts Spades Diamonds), and has a numerical value three greater than the preceding card.(William Coffrin,1898)

2. LITERATURE REVIEW

2.1 Si Stebbins stack (SSS) and Listening Card Trick (LCT): principles and applications

The secret of LCT lies in the magician's ability to determine what the nth card is when he 'listens' to the stack of cards indicated by the blue arrow in Figure 1. Due to the SSS order, the magician can swiftly identify the value and suit of the next nth card using a straightforward formula. Thus, performing this magic trick equates to solving the following problem: Given a sequence $\{a_n\}$ and knowing a_{n-1}, to find the value of a_n (簡煥森 Jian, 2022).

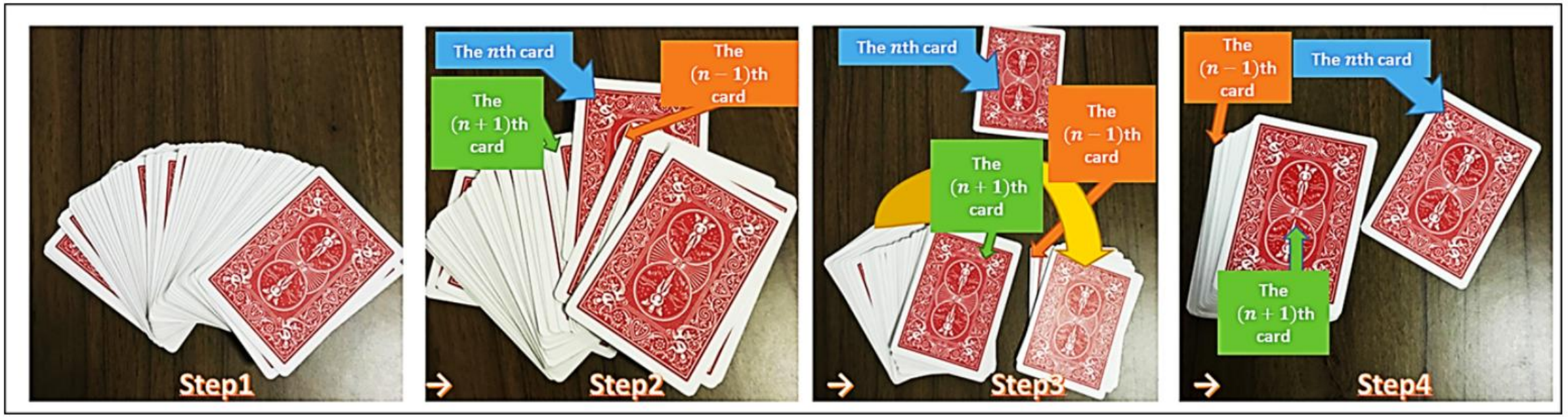


Figure 1: Cut cards and select one

數學建模的重要性：

- Mathematical modeling is of great significance and role in high school mathematics education, as it can cultivate students' **mathematical thinking, creativity, communication, and problem-solving abilities** (Blum & Niss, 1991).

數學建模的五項子能力：

- Mathematical modeling competence consists of five sub-competencies: **structuring situations, mathematizing, processing models, translating, and verifying** (Blum & Kaiser, 2015).

建模水準的過程性評價與結果性評價：

- This study adopts two assessment tools: process-oriented assessment and product-oriented assessment. These tools were developed by Lu et al. (2014), who conducted a study on the assessment of students' mathematical modeling competence based on the analysis of 1357 student test papers and the comparison of Chinese and American curriculum standards.

(魯小莉等，2014)

2. LITERATURE REVIEW

2.2 Mathematical Modeling Competence and Its Assessment

- Process-oriented assessment: This tool focuses on which modeling steps students have experienced in the modeling test, such as identifying variables, making assumptions, verifying models, etc. The authors formed a five-category coding scheme to judge whether students have experienced the key modeling steps.

- Product-oriented assessment: This tool focuses on which key abilities or knowledge levels students have shown in promoting modeling progress, such as structuring situations, mathematizing, processing models, translating, verifying, etc. The authors formed a six-level coding scheme to characterize the modeling ability level that students have reached.

表 1　学生经历的建模步骤

编码	描述	“拉面有多长”的学生解答
类别 1	没有写任何内容，或者只呈现一个结果	空白 14
类别 2	识别变量、作出假设	解，先算出拉了多少根再乘每根长度
类别 3	只给出数学解答，没有识别变量、作出假设	解：$2^7=128$（根） 答：共有 128 根
类别 4	识别变量、作出假设，并获得数学解答	解：假设拉面（人）1.7 米 一根面长＝双臂展开长＝身高≈1.7 米 面条根数为：$2^7=128$（根） 总长度为：$128\times1.7=217.7$（米） 答：总长度约为 217.6 米
类别 5	识别变量、作出假设、获得数学解答、验证模型	不适用

表 3　学生达到的建模能力水平

编码	描述	“拉面有多长”的学生解答
水平 0	无法从实际情境中识别出任何数量关系，无内容，或不相关、无意义内容	空白 14
水平 1	尝试将实际情境结构化、提出问题，但无法找到数学模型，例如文字叙述某些变量、变量间关系	大概 1.7 米 因为一个男性成人身高起码 1.7 米，张开手臂在身高长度上下
水平 2	提出合理的假设，并找到数学模型，但数学模型不合理	$2\times7\times2=28$（米）
水平 3	找到现实模型，转化为合理的数学模型，但未能得到准确的数学解答或数学解答过程错误	人的两手完全张开是 1 米多，图中未完全张开，所以假设是 1 米，对折 7 次 1：1＋1＝2（米）， 2：2＋2＝4（米）， 3：4＋4＝8（米）， 4：8＋8＝16（米）， 5：16＋16＝32（米）， 6：32＋32＝74（米）， 7：74＋74＝148（米） 所以是 148 米左右
水平 4	提出合理的数学模型，得到准确的解答，但没有从实际情境解释结果	一个普通成年人手臂伸开的长度约为人的高度 1.7 米 所以，$1.7\times2^7=1.7\times128=217.6$ 答：总长约为 217.6 米
水平 5	找到现实模型，转化为数学模型，得到准确解答，结合实际情境解释并检验解答、评价数学模型的合理性	不适用

2. LITERATURE REVIEW

2.3 Theoretical Framework of the Study

研究假設:
使用聽牌術作為數學建模的教學情境，可以提高高中生的數學建模能力。

Hypothesis:
Using LCT as a scenario for Mathematical modeling can improve the Mathematical modeling competence of high school students.

The variables of this study are:

- Independent variable: Using LCT as a scenario for Mathematical modeling. This variable refers to **whether** the students **participate** in an elective course on Mathematical modeling based on LCT or not. The students who participate in the course are exposed to LCT as a problem-solving tool and a learning context, while the students who do not participate in the course follow their regular mathematics curriculum.

- Dependent variable: The Mathematical modeling competence of high school students. This variable refers to the comprehensive ability of the students to use knowledge, skills, attitudes, and values in the process of Mathematical modeling. This variable is measured by these assessment tools: **process-oriented assessment** and **product-oriented assessment**, which are based on the five sub-competencies and six levels of Mathematical modeling competence proposed by Blum and Kaiser (2015) and Lu et al. (2014), and a test on **permutation and combination**.

3. RESEARCH DESIGN AND METHOD

3.1 Participants

The research participants were 30 students from grade 11 in a government-aided school in Macau. They were divided into two groups: an experimental group and a control group.

The experimental group consisted of 15 students who voluntary to take an elective course on Mathematical modeling using LCT as a scenario. The course was offered as an optional module consisting of four lessons. The course met twice a week for two hours each time. The students in this group had different levels of prior knowledge and skills on Mathematical modeling, permutation and combination, and card tricks.

The control group consisted of another 15 students who did not take the elective course on Mathematical modeling using LCT as a scenario. They followed their regular mathematics curriculum, which did not include any topics or activities related to Mathematical modeling, or card tricks. The students in this group had similar levels of prior knowledge and skills on these topics as the students in the experimental group.

3.2 Instruments

The Mathematical modeling competence scale：
This scale consists of two parts: a process-oriented assessment and a product-oriented assessment. Each part has five items, each scored on a 5-point Likert scale from 1 (strongly disagree) to 5 (strongly agree). The scale's total score ranges from 10 to 50, with higher scores indicating stronger mathematical modeling competence. This 10-50 score is the **teacher's evaluation after the students' rating**.

The permutation and combination monthly test was a standardized test that measured the students' knowledge level on permutation and combination. The test had 10 multiple-choice questions, each worth 10 points. The total score ranged from 0 to 100, with higher scores indicating greater mastery and application of permutation and combination knowledge.

自編數學建模能力量表:

The Mathematical modeling competence scale was based on the two assessment tools developed by Lu et al. (2014) , with some modifications to suit the context of LCT. The scale consisted of two parts: process-oriented assessment and product-oriented assessment.

- The process-oriented assessment part contained five items, each corresponding to one category of coding scheme that judged whether students have experienced the key modeling steps using LCT.
- The product-oriented assessment part contained five items, each corresponding to one sub-competence of Mathematical modeling using LCT.

Each item was rated on a 5-point Likert scale from 1 (strongly disagree) to 5 (strongly agree).

Process-oriented assessment	Product-oriented assessment
1. I identified variables and made assumptions for LCT.	6. I structured the situation of LCT into a mathematical problem.
2. I used a formula to calculate the cards for LCT.	7. I mathematized the situation of LCT using modular arithmetic and combinatorial permutations.
3. I explained or verified the meaning of the card number or card face in the situation of LCT.	8. I processed the model of LCT using mathematical methods and tools.
4. I analyzed or evaluated the advantages and disadvantages of the model of LCT.	9. I can perform my own magic using the mathematical model of LCT.
5. I reflected on or improved the model of LCT based on different situations or conditions.	10. I am aware of the limitations of LCT magic, and I know how to flexibly apply the mathematical model of LCT when performing magic.

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

3.4.1 TASK SEQUENCE DESIGN (TSD)

問題導向教學
Problem-based Learning (PBL) (Barrows, 1996)

教學模式：提出問題→解決問題→反思過程
Teaching mode: Ask questions →Problem solving → Reflection

特點 Properties

在數學問題的驅動下呈現數學
Mathematics Driven by Mathematical Problems (張奠宙等Zhang et al.,2004)

由問題引起質疑, 圍繞問題進行新課教學
Questioning problems, teaching new courses around them(程厚軍 Cheng,2012)

PBL具有兩大基本特性：問題和探究

2 basic characteristics of PBL:

Questions and Exploration

(陳燕 Chen,2013)

PBL的教學設計步驟
（程厚軍，2012）·

把新知識隱藏於數學問題中

從學生的最近發展區出發創設情境、設置問題

讓學生通過合作、探究 、主動分析 、解決問題

讓學生在此過程中學生親身體驗數學概念 、公式等的形成過程

The Teaching Design Steps Of PBL (Cheng, 2012)

Hide the new knowledge to be learned in mathematical problems

Starting from the students' closest development zone, creating scenarios and setting questions

Allows students to collaborate, explore, proactively analyze, and solve problems

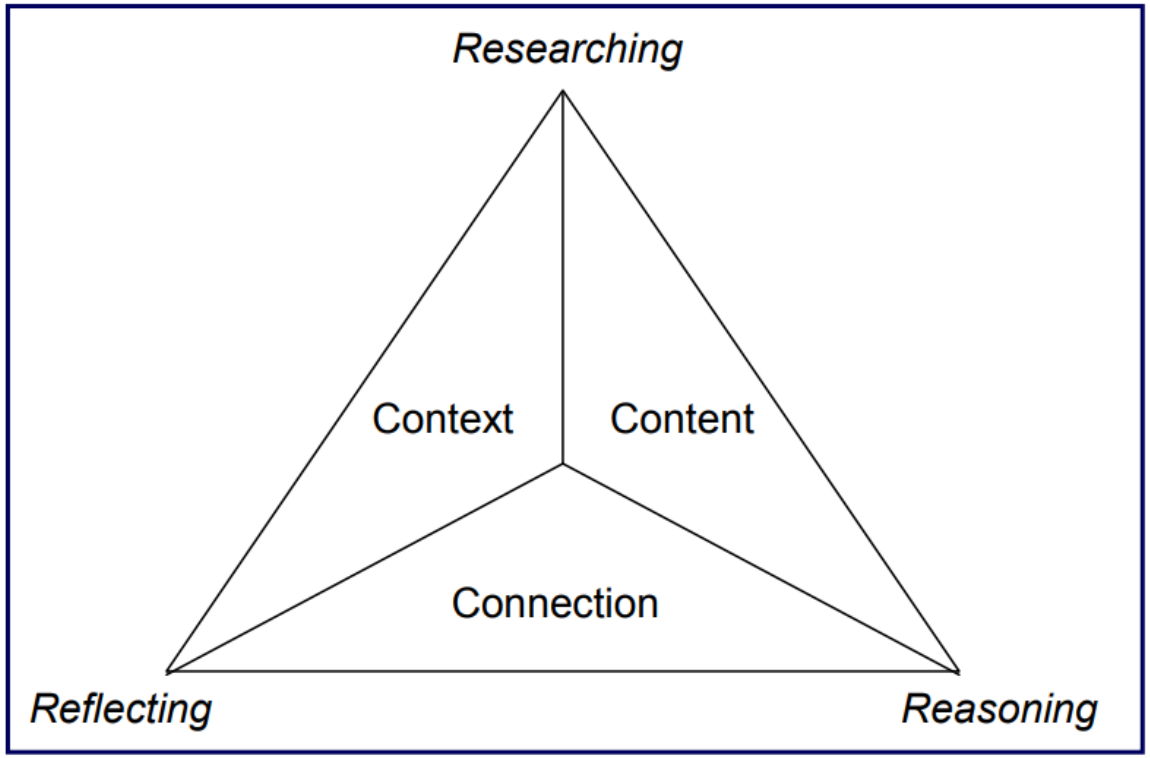
Allowing them to personally experience the formation process of mathematical concepts, formulas, and so on

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

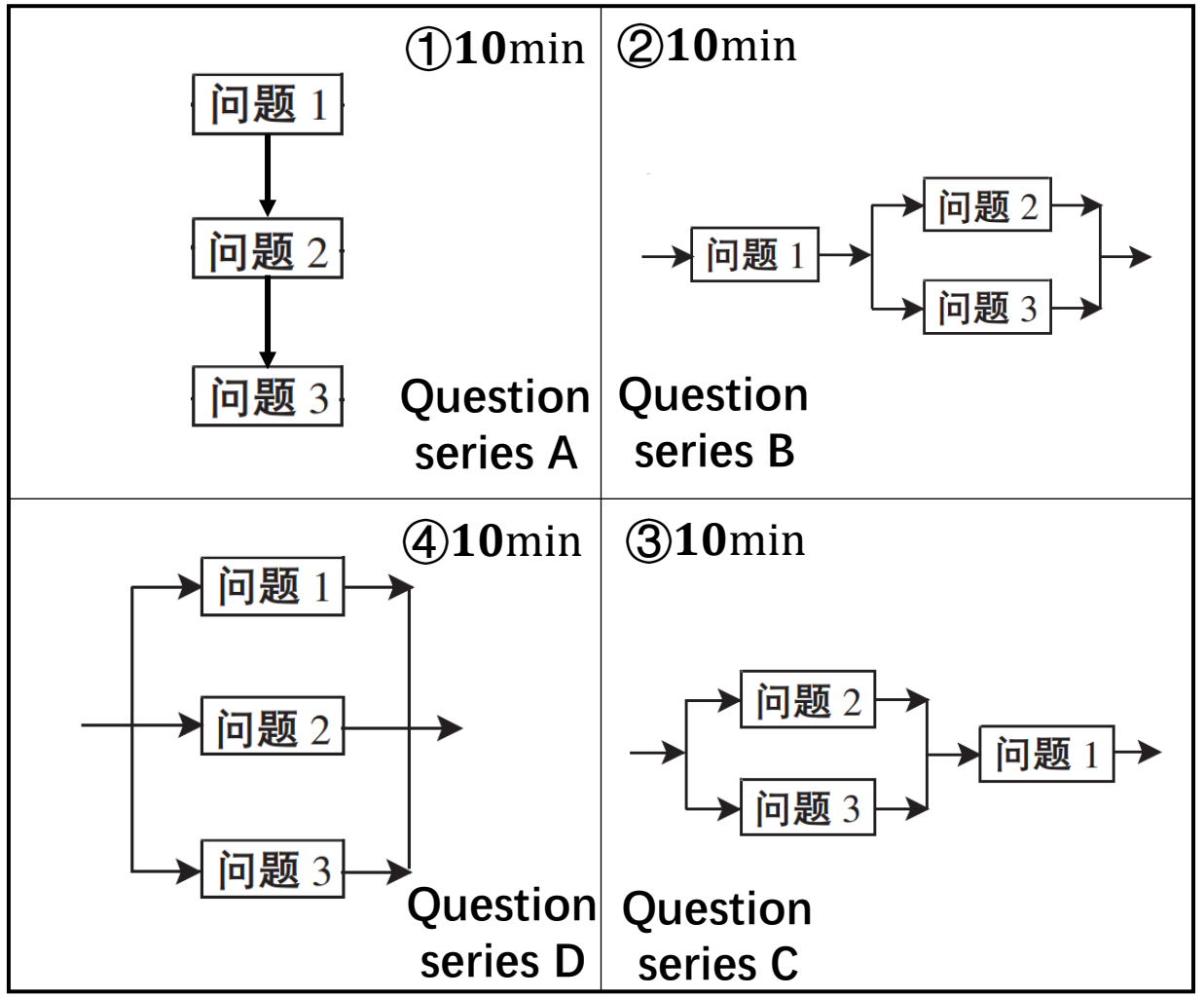
3.4.1 TASK SEQUENCE DESIGN (TSD)

問題選取的3C3R模型
Problem selection：3C3R
(Hung, 2006)



問題驅動課堂教學的基本模式
Problem Driven Classroom
(程厚軍 Chen, 2012)

40min



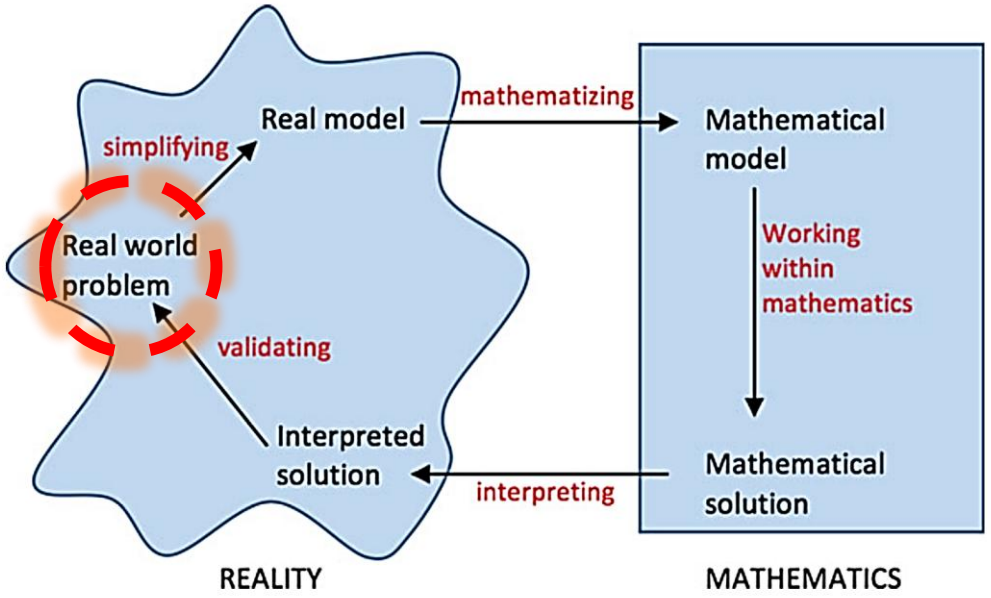
3. RESEARCH DESIGN AND METHOD

3.4 Intervention

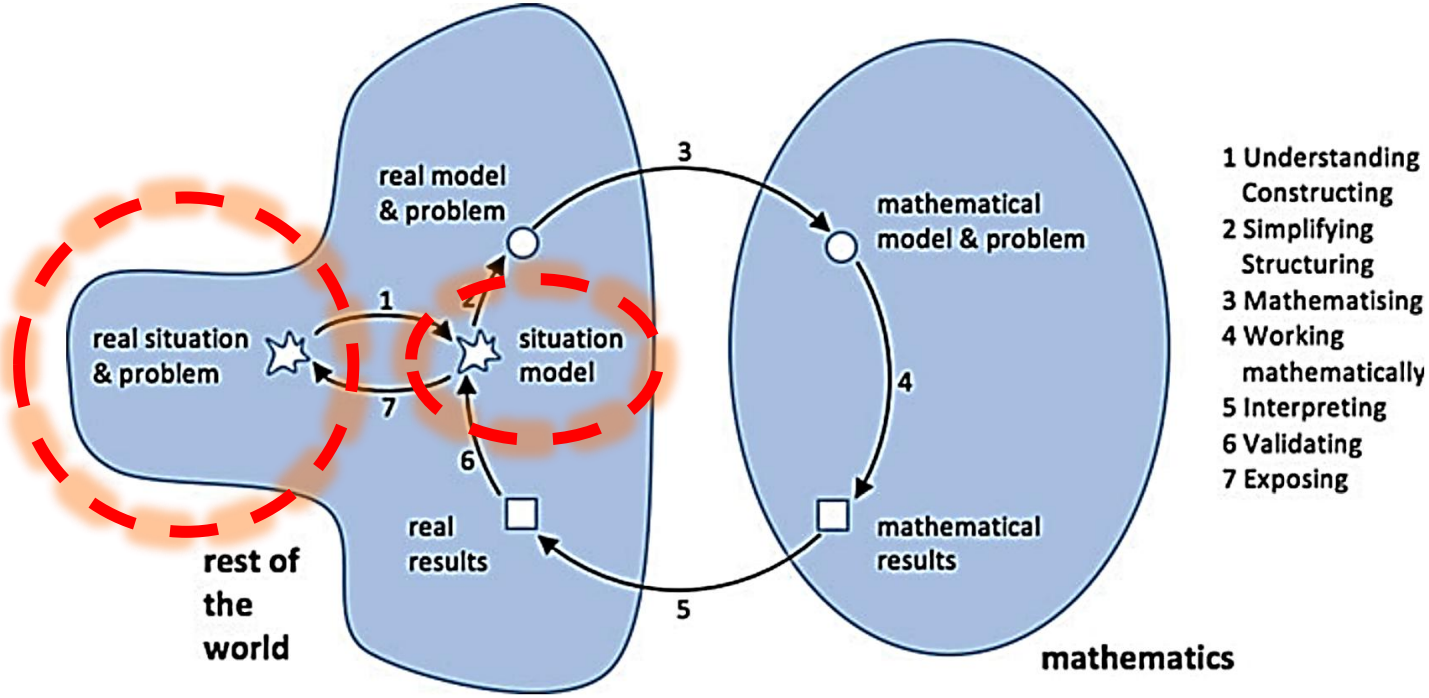
3.4.1 TASK SEQUENCE DESIGN (TSD)

Single mathematising & Complex mathematising

If only one step is used to transfer a real-life problem to a model, this model of a modelling cycle is called **single mathematising**, else is called **Complex mathematising.**(G. Greefrath and K. Vorhölter, 2016)



Single mathematising modeling cycle (SMC)
(Maaß, 2006)

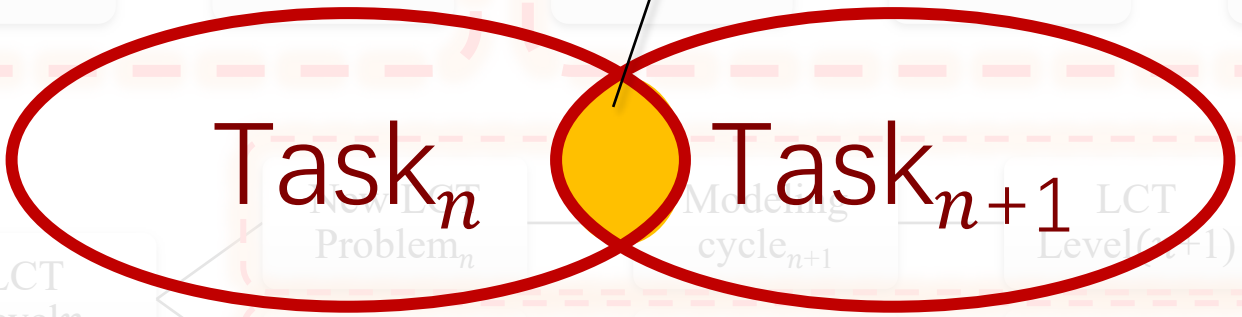


Complex mathematising modeling cycle (CMC)
(Blum and Leiß, 2007)

3. RESEARCH DESIGN AND METHOD

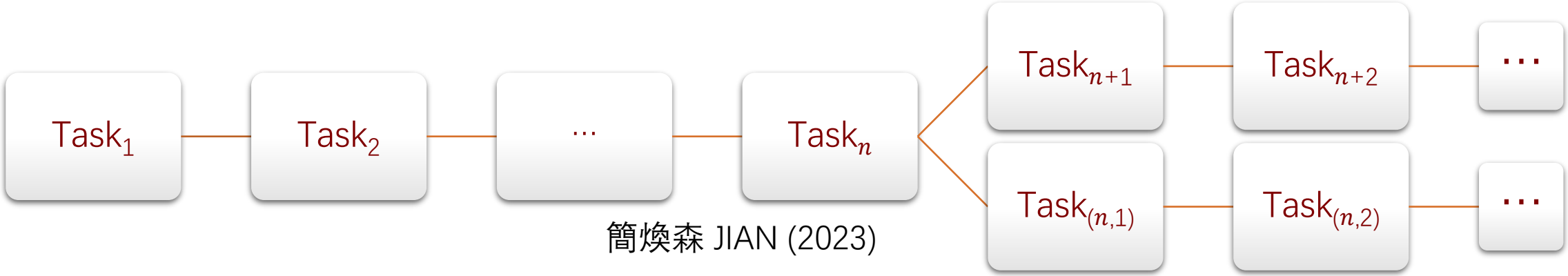
3.4 Intervention

3.4.1 TASK SEQUENCE DESIGN (TSD)



Share the same:

1. Situation
2. Modeling Method
3. ……



簡煥森 JIAN (2023)

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

3.4.2 TSD for MM with LCT

Task$_1$: Find out the first LCT performance formula

Problem: How to convert magic formulas into mathematical formulas?

Magic formulas

Add 3 to the $(n-1)$th card	CHaSeD to derive the pattern

Let x_n represents the number of the nth card, y_n represents the suits of the nth card, and CHaSeD uses 1,2,3,4 to represent respectively.

Mathematical formulas: Formula 1

$x_n \equiv x_{n-1} + 3(mod13)$	$y_n \equiv y_{n-1} + 1(mod4)$

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

3.4.2 TSD for MM with LCT

Task$_2$: Find out more LCT performance formula

Problem: Can I modify Formula 1 by substituting the fixed constants with two non constant parameters? (Formula 2)

$x_n \equiv x_{n-1} + d(mod13)$	$y_n \equiv y_{n-1} + d'(mod4)$

LCT formula detector

A cyclical formula for all 52 cards	d=4,d'=2⇒ ✗	d=4,d'=3⇒ ✓

Dictionary lookup algorithm

Mathematical formulas

24 distinct pairs of and values that convert Formula 2 into LCT sequences.

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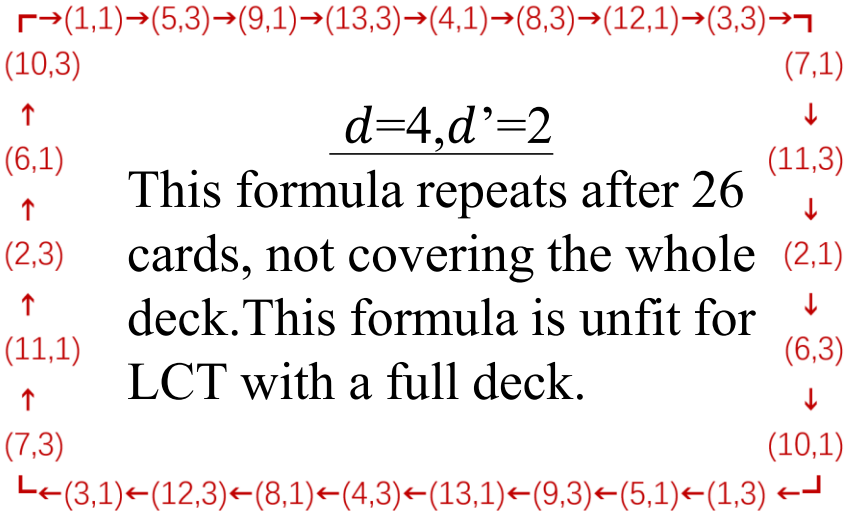
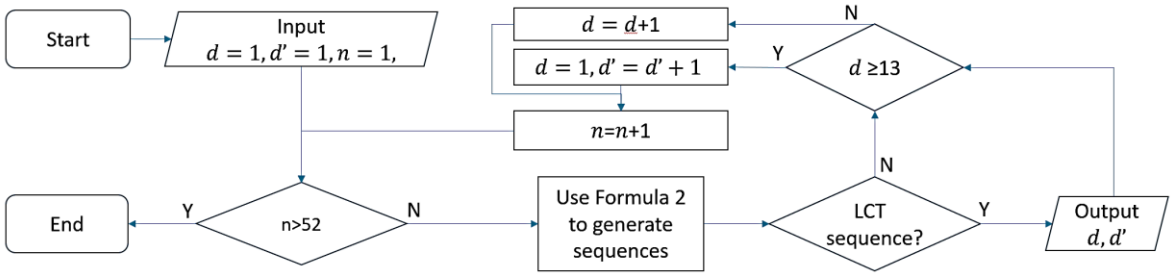


LCT formula detector



Dictionary lookup algorithm

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

3.4.2 TSD for MM with LCT

Task$_3$: Continue to find out more LCT performance formula

Problem: Can I add more non constant parameters in LCT performance formula? (Formula 3)

$x_n \equiv kx_{n-1} + d(mod13)$①	$y_n \equiv k'y_{n-1} + d'(mod4)$②

Try to mix ①,② together to find more non constant parameters (Formula 4)

$x_n \equiv bx_{n-1} + cy_{n-1} + d(mod13)$	$y_n \equiv b'x_{n-1} + c'y_{n-1} + d'(mod4)$

LCT formula detector

Dictionary lookup algorithm

Try $13^3 \times 4^3 = 140{,}608$ times to find LCT sequences

Mathematical formulas

1320 distinct pairs of and values tha convert Formula 4 into LCT sequences.

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

3.4.2 TSD for MM with LCT

Task$_4$: Try to find out perfect LCT performance formula

Problem: How to fix the suit presentation pattern is strictly fixed issue?

Clubs, Hearts, Spades, Diamonds	Diamonds, Spades, Hearts, Clubs

Formula 5: $x_n \equiv bx_{n-1} + cy_{n-1} + d(mod13)$ (Keep formula 4 ① **unchanged**)

$y_n \equiv b'\boldsymbol{x_n} + c'y_{n-1} + d'(mod4)$ (Simply **change** formula 4 ②)

LCT formula detector

Dictionary lookup algorithm

Try $13^3 \times 4^3 = 140{,}608$ times to find LCT sequences

Conclusion: 2084 LCT performance formula

Specifically, 764 perfect LCT performance formula

LCT formula detector

Dictionary lookup algorithm

Try $13^3 \times 4^3 = 140{,}608$ times to find LCT sequences

Conclusion: 2084 LCT performance formula

Specifically, 764 perfect LCT performance formula

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

3.4.2 TSD for MM with LCT

Task$_6$: Continue to find out more LCT performance formula

Problem: Can I add more non constant parameters in LCT performance formula?

Formula 7: $x_n \equiv ax_{n-1} + by_{n-1} + cy_n + d(mod13)$

$y_n \equiv a'x_{n-1} + b'y_{n-1} + d'(mod4)$

Formula 8: $x_n \equiv ax_{n-1} + cy_{n-1} + d(mod13)$

$y_n \equiv a'x_{n-1} + b'x_n + c'y_{n-1} + d'(mod4)$

LCT formula detector

Dictionary lookup algorithm

Try 2,390,336 times to find LCT sequences

Conclusion: 18,320 LCT performance formula

But no new perfect LCT performance formula

The number of LCT sequences

Formula	4	5	6	7	8	Total
LCT Sequences Num.	1320	2084	2084	15740	1052	18112
	The sum of above: 4168			The sum of above:16792		
Perfect LCT Sequences Num.	0	764	764	764	764	1528

3. RESEARCH DESIGN AND METHOD

3.4 Intervention

3.4.2 TSD for MM with LCT

$Task_{(4,1)}$: Evaluating the chaos degree of LCT sequences

Problem: How to evaluate so many LCT performance formula?

Chaos degree : Ascending sequences

The original numbers of the cards range from 1 to 52, forming **a single** ascending sequence.

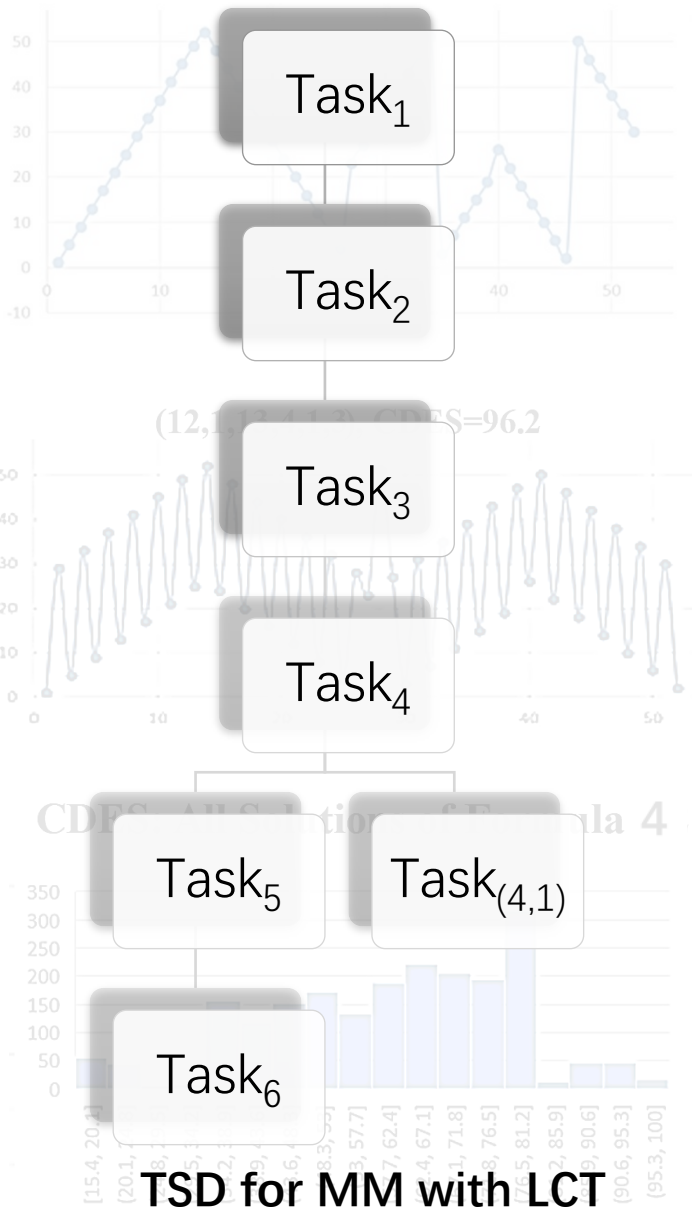
The maximum number of ascending sequences for completely chaotic cards is **26**.

The chaos degree evaluation score(CDES)

CDES= The number of ascending sequences ÷ 26 ×100

In formula 4: CDES(12,7,1,4,1,3) ≈15.4

CDES(12,1,13,4,1,3) ≈96.2



TSD for MM with LCT

CONCLUSION

1. The study employs a task sequence design to scaffold the students' math modelling process with card tricks.
2. The findings show that the students could create their own LCT sequences and perform card magic, and that they enhanced their social interaction and communication in mathematics learning, as well as their embodied cognition, by using their hands and eyes to explore.
3. The study emphasizes the value of card tricks in math modelling education and advocates for the application of task sequence design in teaching.
4. The study also proposes further research on how to incorporate this card trick into relevant chapters of teaching materials, such as congruence equations, permutations and combinations, and algorithm design, to offer stimulating supplementary reading for students.

REFERENCES

Jian, H. (2023). *Card Tricks As A Context For Mathematical Modelling: A Task Sequence Design And Implementation*[Paper presentation]. ICME-15, Sydney, Australia

Jian, H. (2022). The mathematical modelling of listening card trick. In H. Zhu(Eds.), *Mathematical modelling teaching design and case collection* (pp. 91-109). Beijing, China: China Renmin University Press

AWARDS (RELATED TO LCT)

Jian H. (2022). The action research on the school-based course of mathematical modeling using "Mathematical Model of Listening to Cards" as an example. Category: Pedagogical Research. Awards: Class A*. Education and Youth Development Bureau of Macau

Jian H. (2022). Magic Card Sequence Generator. Category: Teaching Aids. Class B*. Education and Youth Development Bureau of Macau

Jian H. (2022). Exploring General Term Formulas for Sequences. Category: Lesson Plan. Class A*. Education and Youth Development Bureau of Macau

**There are three types of ranking awards for all categories: Distinction, Class A, Class B.*